Mathematics for *Navigators*

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Devoted exclusively to mathematical preparation for the study of navigation, this book reviews addition, subtraction, [illegible] arithms, plane trigonometry, and spherical [illegible], for those who have previously studied these [illegible]

[illegible] taking up navigation without previous training [illegible] ches of mathematics, the book is sufficiently [illegible] e to enable them to approach navigation [illegible] gh understanding of the mathematics involved.

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CHAPTER II

ALGEBRA

Definitions and Terms. Use of Parentheses and Brackets. Powers, Exponents, and Roots. Equations. Time, Speed, and Distance. Propeller Pitch and Slip.

DEFINITIONS AND TERMS

Algebra is that branch of mathematics which treats of the relations and properties of quantity by means of letters and other symbols. It is applicable to those relations which hold good for every kind of magnitude. Algebra is concerned only with addition, subtraction, multiplication, divisions, involutions, and the extraction of roots, of finite numbers. It is used primarily in the construction and solution of equations.

Known quantities are usually represented by letters from the beginning of the alphabet (a, b, c, d, etc.), and unknown quantities by letters from the end of the alphabet (x, y, z, etc.). Quantities may be positive (+) or negative (−). The negative sign is always indicated, but the positive sign is generally omitted from the first quantity of an expression. When no sign is shown, the (+) is understood.

Some symbols and arbitrary signs commonly used in algebra, besides those which are mentioned elsewhere, are

$>$ is greater than	$\neq$ is not equal to
$<$ is less than	∞ infinity

$\sim$ difference, meaning algebraic difference. For example, the difference between plus 5 and minus 4 is 9. The difference between 20 degrees *north* and 10 degrees *south* is 30 degrees, whereas the difference between 20 degrees *north* and 10 degrees *north* is only 10 degrees.

Addition, or the sum of two or more quantities, is represented by the *plus* or addition sign (+). For example, the sum of a and b is indicated by the expression $a + b$. The sum of a, b, and x is $a + b + x$. In addition, negative numbers keep their negative or minus signs. For example, the sum of a and $(-)\,b$ is $a - b$.

Subtraction, or the difference between quantities, is represented by the *minus* or subtraction sign (−). For example, b subtracted from a is indicated by the expression $a - b$. The expression $a - b - c$ means that from a you must subtract b, and then subtract c from what is left.

Subtraction of a negative quantity is the same as addition. As in grammar, two negatives make an affirmative. Thus, minus $(-)\,4$ is the same as *plus* 4. For example, $a - (-b) = a + b$.

Multiplication, or the product of two or more quantities, is represented by the multiplication sign (×). Thus, the product of 4 times 2 is indicated as 4×2. When letters are used in place of numerals, the product may be indicated without the multiplication sign. For example, a multiplied by b, instead of being written as $a \times b$ may be expressed simply as ab. Also, 4 times c may be written as $4c$.

Division, or the process of dividing one quantity by another, is indicated by the division sign (÷).

The *dividend* is the number that is to be *divided*, and the *divisor* is the number by which it is to be divided. Division may be indicated by the division sign (÷) or by a fraction in which the *numerator* is the dividend, and the *denominator* the divisor. For instance, a divided by b may be expressed as $a \div b$, $\frac{a}{b}$, or a/b.

USE OF PARENTHESES AND BRACKETS

Parentheses ( ), brackets [ ], and braces { } are used to enclose or group together two or more quantities that are to be treated as a unit. Thus, $a - (b + c)$ means that from a the sum of b and c is to be subtracted. The use of these symbols frequently permits a complicated expression to be simplified.

It is also frequently necessary to represent an expression with the parentheses and brackets eliminated.

Suppose that from a we wish to subtract b and then c. This can be represented by the expression $a - b - c$. Now, the same result will be achieved if we first add b and c, and then subtract their *sum* from a, as $a - (b + c)$.

If we want to add a and b and subtract c from their sum, this may be expressed as $a + b - c$. The same answer would be obtained if we first took c from b and then added the remainder to a, as, $a + (b - c)$.

These two examples may be expressed in the form of two equations as follows:

$$a - b - c = a - (b + c) \tag{1}$$
$$a + b - c = a + (b - c) \tag{2}$$

It will be noted that if parentheses are begun after a *minus* $(-)$ sign, the sign of each quantity inside the parentheses must be changed. However, when the parentheses are introduced after a *plus* sign $(+)$, the signs of the enclosed quantities remain the same.

When a quantity immediately precedes a parenthetical expression (with no plus or minus sign between them), it indicates that every term within the parentheses is to be multiplied by that quantity. Thus, $a(b - c) = ab - ac$. The expression $ab - ac$ is said to be *simplified* when it is expressed as $a(b - c)$.

Example: Simplify $ax - ay - bd + be$
Ans. $a(x - y) - b(d - e)$

The general principles of the foregoing operations should be understood, but the study of navigation does not require the solution of complicated problems of this nature.

If further simplification is necessary after the use of parentheses, brackets are used, and the same principles apply. Thus, $a[b(c + d) + e(x - y)] = abc + abd + aex - aey$. The former is a *simplified* expression of the latter.

EXERCISE

1. Simplify the following expressions, using parentheses and brackets as necessary:

(*a*) $ab + ac - xy + xz$
(*b*) $abc - acd - axy - axz + t$

Ans. (*a*) $a(b + c) - x(y - z)$
(*b*) $a[c(b - d) - x(y + z)] + t$

2. Represent the following expressions with brackets and parentheses eliminated. Eliminate parentheses first.

(*a*) $b[c(a - d + e) - n(pq - rs) + z] - y$.
(*b*) $a(b + c) - e[x(y - z) + t - m]$.

Ans. (*a*) $bca - bcd + bce - bnpq + bnrs + bz - y$
(*b*) $ab + ac - exy + exz - et + em$

POWERS, EXPONENTS, AND ROOTS

If a quantity is to be multiplied by itself a number of times, this is called *raising it to a power*, the *power* being the number of equal factors. Thus, $a \times a \times a \times a$, or $aaaa$, indicates raising a to the fourth power. This process may be indicated by the use of an *exponent* which is a number placed above and to the right to indicate the power. Thus, $aaaa$ may be expressed as a^4, which is read "a to the fourth power."

The exponent 1 is never used but is understood in the absence of any other exponent. The expression b^2 is read "b squared," because if the side of a square is b, its area is b^2. Likewise, d^3 is read "d cubed," for if the side of a cube is d, its volume is d^3.

A positive quantity, raised to any power, remains positive. A negative quantity changes sign with each raise, becoming positive for even powers and negative for odd powers, in accordance with the rule that "two (or 4, or 6, etc.) negatives make an affirmative." Thus $(-b)^2 = b^2$, $(-b)^3 = -b^3$, $(-b)^4 = b^4$, etc.

Multiplication of like terms can be accomplished by adding the exponents. Thus, aaa becomes a to the $1 + 1 + 1$ power, or a^3. Likewise, $b^2 \times b^3 = b^5$ and $c \times c^3 = c^4$.

Division of like terms can be accomplished by subtracting the exponent of the divisor from that of the dividend. Thus, $a^5 \div a^3 = \frac{aaaaa}{aaa} = a^{5-3} = a^2$.

A negative exponent can result if the exponent of the divisor is greater than that of the dividend. Thus, $a^3 \div a^5 = \frac{aaa}{aaaaa} = \frac{1}{a^2}$ or it equals a^{3-5}, or a^{-2}. Thus, a *negative* exponent in the numerator becomes *positive* in the denominator, and vice versa.

Example: Express the following with positive exponents only:

$$\frac{a^2x^{-3}}{b^{-4}y^3} \quad \text{Ans. } \frac{a^2b^4}{x^3y^3}$$

The opposite of raising a quantity to a given power is called the *extraction of a given root.* The root symbol is $\sqrt{\ }$, called the *radical sign*, and the desired root is indicated by a numeral placed as follows, $\sqrt[3]{\ }$. If no root number is used, the number 2 is understood. Thus,

$\sqrt{a}$ is read "the square root of a."
$\sqrt[3]{b}$ is read "the cube root of b."
$\sqrt[4]{c}$ is read "the fourth root of c," etc.

In extracting an even root of a positive number, it must be remembered that in multiplying, an even number of *negative* signs results in a *positive* quantity. Thus, both $(+a)^2$ and $(-a)^2$ equal plus a^2. Therefore, the square root of a^2 may be either $(+)\,a$, or $(-)\,a$. To indicate that either plus or minus is correct, the symbol $\pm$ (plus or minus) is used. Thus, $\sqrt{b^2} = \pm b$.

Instead of using the radical sign, fractional exponents may be employed. For example, $\sqrt{x}$ means the same as $x^{\frac{1}{2}}$, and $\sqrt[3]{x^2}$ is equivalent to $x^{\frac{2}{3}}$.

For $$\sqrt{x} \times \sqrt{x} = \sqrt{x^2} = x$$
and $$x^{\frac{1}{2}} \times x^{\frac{1}{2}} = x^{\frac{1}{2}+\frac{1}{2}} = x$$

When the radical sign is removed, the *exponent* of the expression *under* it must be *divided* by the indicated root. Thus, $\sqrt[4]{a^3} = a^{\frac{3}{4}}$; $\sqrt[3]{a^2} = a^{\frac{2}{3}}$, etc.

5. Like powers of equals are equal.

Examples: (a) If $a = b$, then $a^4 = b^4$

(b) If $a = b + c$, then $a^2 = (b + c)^{2}$* $= b^2 + 2bc + c^2$

* Be careful to raise $(b + c)$ *collectively*, not $b^2 + c^2$.

6. Like roots of equals are equal.

Examples: (a) If $a = b$, then $\sqrt[3]{a} = \sqrt[3]{b}$

(b) If $a = b + c$, then $\sqrt{a} = \sqrt{b + c}$

(c) $25 = 16 + 9$, $\sqrt{25} = \sqrt{16 + 9}$*

* *Not* $\sqrt{16} + \sqrt{9}$.

Some other operations commonly employed in rearranging the terms of an equation to facilitate solution are, in reality, merely simplified applications of some of the foregoing axioms.

1. Transposing a term from one side of the equation to the other and changing its sign. For example, if $x + a = b + c$ and we wish to obtain an expression for x in terms of a, b, and c, we may *transpose* the a to the other side of the equal sign and change its sign, obtaining $x = b + c - a$. This is equivalent to the axiom, "If equals are subtracted from equals the remainders are equal." For, if $x + a = b + c$, then, by subtracting $a = a$, we get $x = b + c - a$.

Likewise, transposing a negative term from one side to the other and changing its sign to positive is equivalent to adding equals to equals.

2. Cross multiplying, or multiplying each numerator by the denominator of the opposite side of the equation. For example, if $\frac{a}{b} = \frac{c}{d}$, then $ad = cb$. This is the same as multiplying each side of the original equation by bd, or $\frac{a}{b} \times bd = \frac{c}{d} \times bd$. By cancellation, there remains $ad = cb$. Cross multiplying, therefore, is the same as multiplying equals by equals.

It is important for the student to be able to arrange an equation in such a manner as to indicate the solution for the desired quantity. This requires that the unknown quantity alone be on the left side of the equation, and that all the known quantities from which it is to be found be properly arranged on the right side of the equation.

EXERCISE

4. Arrange the following equations to indicate the solution for x.

(*a*) $x + a = 2x - b$ *Ans.* (*a*) $x = a + b$

(*b*) $ax + c = bx - d$ (*b*) $x = \dfrac{c+d}{b-a}$

(*c*) $ax - b = x + c$ (*c*) $x = \dfrac{c+b}{a-1}$

(*d*) $ax - bc = bd - 2x$ (*d*) $x = \dfrac{b(c+d)}{a+2}$

Problems in algebra are generally solved by letting a letter stand for an unknown quantity. Then, from the given data, one or more equations are set up and the unknown quantity is determined.

Example: The sum of two numbers is 15, and one is 3 greater than the other. What are the numbers?

Solution: Let $x =$ the smaller number
then $x + 3 =$ the larger number
and $x + x + 3 = 15$ (the given sum)
or $2x = 15 - 3$
$x = 6$

Ans. The numbers are 6 and 9.

EXERCISES

5. A man is twice as old as his son, and the sum of their ages is 81 years. How old is each? *Ans.* 54 and 27.

6. The sum of two numbers is 27, and their difference is 3. What are the numbers? *Ans.* 15 and 12.

7. One-half of a certain number is added to four-fifths of the same number, and the sum is 39. What is the number?

Solution: Let $x =$ the number

then $$\frac{x}{2} + \frac{4x}{5} = 39$$

Now multiply *each term* by a number such that the 2 and 5 will disappear. This number is 10. Then we get

$$5x + 8x = 390$$
$$13x = 390$$
$$x = 30$$

8. Three-fourths of a certain number is 10 more than one-third of the same number. What is the number? *Ans.* 24.

9. A man is now four times as old as his son. In 14 years he will be only twice as old as his son. How old are they now?

Ans. 28 and 7.

TIME, SPEED, AND DISTANCE

The unit of distance employed in navigation is the *nautical mile*. This is the length of 1′ of arc of a great circle on the earth. The nautical mile measures about 6,080.3 ft. This value will be used in the examples and exercises here.

The unit of speed used in navigation is the *knot*. A knot is a speed of 1 nautical mile per hour. Since the knot is a unit of speed and not of distance, it is correct to say that a vessel is making 20 knots. In marine navigation distances are rounded off to tenths of mile, and speeds to tenths of a knot. In problems involving time, speed, and distance, time is generally rounded off to the nearest whole minute. If any two of the quantities are given, the third is readily computed.

Example 1: A vessel travels 325 miles in a day. What was her average speed in knots?

Solution:

$$\text{Speed} = \frac{\text{distance}}{\text{time}} = \frac{325}{24} = 13.5 \text{ knots}$$

Example 2: A vessel is steaming at 18 knots. How far will she travel in $9^h 30^m$?

Solution:

$$\text{Distance} = \text{speed} \times \text{time} = 18 \times 9.5 = 171 \text{ miles}$$

Example 3: A vessel is to make a trip of 133 miles. How long will it require, at 17 knots?

Solution:

$$\text{Time} = \frac{\text{distance}}{\text{speed}} = \frac{133}{17} = 7.82^h = 7^h\,49^m$$

EXERCISE

10. In the following problems, two of the three quantities are given. Compute the unknown quantities, rounding off speeds and distances to the nearest tenth, and times to the nearest minute.

	Time	Speed, knots	Distance, miles	Answers
(*a*)	$19^h\,00^m$	17.4	?	330.6 miles
(*b*)	$1^d\,11^h\,30^m$	22.0	?	781.0 miles
(*c*)	$8^h\,18^m$	?	135.4	16.3 knots
(*d*)	$1^d\,15^h\,00^m$	?	546.2	14.0 knots
(*e*)	?	18.5	723.0	$1^d\,15^h\,04^m$
(*f*)	?	11.0	183.7	$16^h\,42^m$

PROPELLER PITCH AND SLIP

The *pitch* of a propeller is the distance it would advance in one revolution if it were rotated in an unyielding medium. Pitch is generally expressed in feet.

The difference between the distance actually traveled by a ship and the distance that would have been traveled if the propeller were operating in an unyielding medium is called *slip*. Slip is generally expressed as a *percentage* of this difference in relation to the theoretical distance. If the theoretical distance is T and the actual distance is A, then slip is $(T - A)/T$.

It will be noted that as long as A is less than T, the slip will be *positive*, but if A is greater than T the slip will be *negative*. The propeller itself cannot cause A to exceed T. However, when aided by favorable winds or currents, it sometimes happens that A will exceed T, with a resultant negative slip. Unless specifically mentioned to the contrary, the slip is assumed to be positive.

Slip may also be considered in relation to *speed*. In this case, it is equal to the theoretical speed (S_T), minus the actual speed

(S_A), divided by the theoretical speed (S_T) or,

$$\text{slip} = \frac{(S_T - S_A)}{S_T}.$$

In solving problems involving slip, the nautical mile is taken as 6080.3 ft. The pitch of the propeller is a known quantity, as is the number of revolutions per minute. If the actual speed or distance made good is known, the slip is readily computed. Likewise, if the slip is known, the actual speed or distance may be determined.

The slip will vary considerably with different drafts for the same vessel, and also with the propeller speed, being *greater* for *shallow draft* and *high speed.*

Naval vessels are generally *calibrated* for speed in terms of revolutions per minute, since the draft is not subject to wide variation, as is that of a cargo vessel. This calibration is accomplished during trials, by running back and forth past a measured mile at various propeller speeds. The speed of the ship is determined by dividing the number of minutes required to go the mile, into 60.

Example 1: How far will a vessel go in 1^h if the pitch of the propeller is 18 ft., making 100 revolutions per minute (r.p.m.), if there were no slip? (Answer to be in nautical miles to the nearest .1 .)

Solution:

$$d = \frac{18 \times 100 \times 60}{6080.3} = 17.8$$

Example 2: If the vessel makes good a distance of 16.2 miles during this hour, what is the slip (to the nearest 1 per cent)?

Solution:

$$\text{Slip} = \frac{17.8 - 16.2}{17.8} = .09, \text{ or } 9 \text{ per cent}$$

The *efficiency* of the propeller may be considered as 100 per cent, minus the slip. In the foregoing example, the efficiency of the propeller would be 91 per cent.

Example 3: The pitch of a propeller is 20 ft.; r.p.m., 90; slip, 3 per cent. What is the actual speed of the vessel?

Solution:

$$S_A = \frac{20 \times 90 \times 60 \times .97}{6080.3}$$
$$= 17.2 \text{ knots}$$

EXERCISES

Solve the following problems involving propeller pitch, slip, speed, and distance:

11. Given: Pitch, 15.8 ft.; r.p.m., 88; slip, 6 per cent.

Determine: Speed made good (knots to .1).

Ans. 12.9.

12. Given: Pitch = 20.4 ft.; r.p.m. = 100; speed made good = 19.0 knots.

Determine: Slip to nearest 1 per cent.

Ans. 6 per cent.

13. Determine the missing quantities:

	Pitch, feet	R.p.m.	Slip, per cent	Speed made good, knots	Answers
(*a*)	16.5	104	6	?	15.9 knots
(*b*)	18.0	98	3	?	16.9 knots
(*c*)	17.7	110	(−) 2	?	19.6 knots
(*d*)	19.3	80	?	14.0	8 per cent
(*e*)	15.0	125	?	16.3	12 per cent
(*f*)	16.7	117	?	18.5	4 per cent

2. A vessel, wishing to check its speed, steams past a measured nautical mile, covering the distance in $2^m\,46^s$. What was its speed in knots? (Round off the answer to .1 knot.) *Ans.* 21.7 knots.

3. A vessel steamed 2178.6 nautical miles in $5^d\,17^h\,39^m$. What was her average speed in knots to the nearest .1? *Ans.* 15.8 knots.

Solve the following problems by using logarithms:

4. $\dfrac{(436.8)^2}{974.3}$ *Ans.* 195.83.

5. $\dfrac{3.685 \times 63.982}{49.75}$ *Ans.* 4.7391.

6. A vessel steams a distance of 1768 nautical miles in $3^d\,17^h\,48^m$. What was her average speed in knots to the nearest .1?

Ans. 19.7 knots.

7. The pitch of a certain propeller is 18 ft. 9 in. At 110 r.p.m. the vessel makes good a distance of 2585.4 nautical miles in $5^d\,11^h\,30^m$. What was the slip of the propeller to the nearest 1 per cent?

Ans. 3 per cent.

8. Find the logarithms of the following numbers:

(*a*) 4826.7	*Ans.* (*a*) 3.68365
(*b*) 3865.8	(*b*) 3.58724
(*c*) 482.62	(*c*) 2.68361
(*d*) 382.47	(*d*) 2.58260
(*e*) 87.393	(*e*) 1.94148
(*f*) 7.2966	(*f*) .86312
(*g*) 3.1416	(*g*) .49715
(*h*) .74835	(*h*) 9.87410 − 10
(*i*) .032987	(*i*) 8.51834 − 10
(*j*) .0063154	(*j*) 7.80040 − 10
(*k*) .00032488	(*k*) 6.51172 − 10
(*l*) .000048296	(*l*) 5.68391 − 10